

A measure of intelligence of an approximation of a real number in a given model

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Abstract

In this paper, we present a way to measure the intelligence (or the interest) of an approximation of a given real number in a given model of approximation. Basing on the idea of the complexity of a number, defined as the number of its digits, we introduce a function noted μ (called a measure of intelligence) associating to any approximation **app** of a given real number in a given model a positive number $\mu(\mathbf{app})$, which characterises the intelligence of that approximation. Precisely, the approximation **app** is intelligent if and only if $\mu(\mathbf{app}) \geq 1$. We illustrate our theory by several numerical examples and also by applying it to the rational model. In such case, we show that it is coherent with the classical rational diophantine approximation. We end the paper by proposing an open problem which asks if any real number can be intelligently approximated in a given model for which it is a limit point.

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1 Introduction

Throughout this paper, we let \mathbb{N}^* denote the set $\mathbb{N} \setminus \{0\}$ of positive integers. We denote by \log the natural logarithm function. On the other hand, for reasons of technical convenience, we denote a finite regular continued fraction

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}$$

simply by $[a_0, a_1, a_2, \dots, a_n]$ and we denote an infinite regular continued fraction

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

by $[a_0, a_1, a_2, \dots]$ (where $a_0, a_1, \dots \in \mathbb{R}$).

If an infinite regular continued fraction $[a_0, a_1, a_2, \dots]$ is periodic, precisely if there exist $k \in \mathbb{N}$ and $d \in \mathbb{N}^*$ such that for any $n \geq k$, we have $a_{n+d} = a_n$, then we denote it (as usually) by

$$[a_0, a_1, a_2, \dots, a_{k-1}, \overline{a_k, a_{k+1}, \dots, a_{k+d-1}}].$$

More than two thousand years ago, mathematicians and calculators were interested to approximate some important real numbers by the values of some type of functions with integer variables. For example the number π is approximated by $\frac{22}{7}$ (by Archimedes) and also by $\sqrt{2} + \sqrt{3}$ and by $\sqrt{\frac{40}{3} - 2\sqrt{3}}$ (by Cholesky). All those approximations are recognized *interesting* (or *intelligent*) but up to now we don't know what is the mathematical sense of the word "interesting"! For example, our intuition tells us that the approximation $\pi \simeq \frac{22}{7}$ is more interesting (or more intelligent) than $\pi \simeq \frac{314159}{100000}$ even if the last approximation is more accurate than the first one. In this particular example, we can explain this preference by the fact that the size of the approximation $\frac{22}{7}$ of π is much less than the size of the approximation $\frac{314159}{100000}$ (where the size of a rational number $\frac{a}{b}$, with $a, b \in \mathbb{Z}$, $b \neq 0$ and $\gcd(a, b) = 1$, can be taken equal to $\max(|a|, |b|)$). An explication more clear is that there is no rational approximation of π which is better than $\frac{22}{7}$ and which has a positive denominator ≤ 7 . A complementary reason is that $\frac{22}{7}$ is a regular continued fraction convergent of π . However, the approximation $\pi \simeq \frac{314159}{100000}$ is naive because we can find a better rational approximation of π with smaller positive denominator (for example $\pi \simeq \frac{355}{113}$). From this example, we see that the *intelligence* (or the interest) of an approximation of a real number is a harmonic melange of the two following characteristics:

- (i) The size of the approximation (or its simplicity).
- (ii) The accuracy of the approximation.

The accuracy of an approximation of a given real number x (say $x \simeq \alpha$) is characterised by its error, which is equal to $|x - \alpha|$. Also the size of an approximation can be easily defined by fixing a *model of approximation* (see below). For example, for the *rational model*, the size of the approximation $x \simeq \frac{p}{q}$ (where $p, q \in \mathbb{Z}$, $q \neq 0$) can be defined by $s(x \simeq \frac{p}{q}) = |pq|$. But what remains vague in the definition of the intelligence of an approximation is the harmony in the melange of the two characteristics (i) and (ii). The objective of this paper is to clarify rigorously this harmony. After that, we define a *measure of intelligence* of an approximation of a given real number (depending on a model) such that:

- If it is < 1 , the approximation is naive;
- If it is ≥ 1 , the approximation is intelligent.

In addition, more the *measure of intelligence* of an approximation is greater more that approximation is intelligent.

2 Models of approximation

Definition 2.1. We call a *model of approximation* any application $\mathfrak{M} : \mathbb{Z}^{*n} \rightarrow \mathbb{R}$ (where n is a positive integer). The variables of \mathfrak{M} are called *parameters* and we say that \mathfrak{M} is a model of n parameters.

Examples 2.2. The following representations:

$$\frac{a}{b} \quad (\mathfrak{M}_1)$$

$$a + b\sqrt{2} \quad (\mathfrak{M}_2)$$

$$a + b\sqrt{c} \quad (\mathfrak{M}_3)$$

$$\frac{a}{b + c \log 2 + d \log 3} \quad (\mathfrak{M}_4)$$

(where $a, b, c, d \in \mathbb{Z}^*$) define models of approximation. We call \mathfrak{M}_1 *the rational model*; so the rational model is a model of two parameters. The model \mathfrak{M}_2 is also of two parameters; while \mathfrak{M}_3 is a model of 3 parameters and \mathfrak{M}_4 is a model of 4 parameters.

Definition 2.3. Let \mathfrak{M} be a model of approximation of n parameters ($n \in \mathbb{N}^*$) and let x be a real number. We say that an approximation of x *belongs* to \mathfrak{M} if it has the form: $x \simeq \mathfrak{M}(a_1, \dots, a_n)$ (where $a_1, \dots, a_n \in \mathbb{Z}^*$).

In the above examples, we remark that any approximation of a given real number x belonging to \mathfrak{M}_2 belongs to \mathfrak{M}_3 . In this case, we say that the model \mathfrak{M}_3 is finer than the model \mathfrak{M}_2 ; or equivalently, the model \mathfrak{M}_2 is coarser than the model \mathfrak{M}_3 . More precisely, we have the following definition:

Definition 2.4. Let \mathfrak{M} and \mathfrak{M}' be two models of approximation and let n and n' be respectively their number of parameters. We say that \mathfrak{M}' is finer than \mathfrak{M} (or equivalently \mathfrak{M} is coarser than \mathfrak{M}') if we have:

$$\mathfrak{M}(\mathbb{Z}^{*n}) \subset \mathfrak{M}'(\mathbb{Z}^{*n'}).$$

3 The size and the logarithmic size of an approximation of a real number belonging to a given model

Definition 3.1. Let \mathfrak{M} be a model of approximation of n parameters ($n \in \mathbb{N}^*$) and let x be a real number. We define *the size* of an approximation $x \simeq \mathfrak{M}(a_1, \dots, a_n)$ of x (belonging to \mathfrak{M})

by:

$$\mathbf{s}(x \simeq \mathfrak{M}(a_1, \dots, a_n)) := |a_1| \times |a_2| \cdots \times |a_n|.$$

— We also define *the logarithmic size* of the same approximation by:

$$\mathbf{s}_{\log}(x \simeq \mathfrak{M}(a_1, \dots, a_n)) := \log \mathbf{s}(x \simeq \mathfrak{M}(a_1, \dots, a_n)) = \log |a_1| + \log |a_2| + \cdots + \log |a_n|.$$

Note that, we have always $\mathbf{s}(x \simeq \mathfrak{M}(a_1, \dots, a_n)) \in \mathbb{N}^*$ and $\mathbf{s}_{\log}(x \simeq \mathfrak{M}(a_1, \dots, a_n)) \in \mathbb{R}^+$.

— If $\mathbf{s}_{\log}(x \simeq \mathfrak{M}(a_1, \dots, a_n)) \neq 0$ (that is, if the a_i 's are not all equal to ± 1), we say that the approximation $x \simeq \mathfrak{M}(a_1, \dots, a_n)$ of x is *admissible*.

4 A measure of intelligence of an approximation of a real number belonging to a given model

Let \mathfrak{M} be a model of approximation of n parameters ($n \geq 1$) and x be a nonzero real number not representable in \mathfrak{M} (i.e., $x \notin \mathfrak{M}(\mathbb{Z}^{*n})$). Let

$$x \simeq \mathfrak{M}(a_1, \dots, a_n) \tag{*}$$

(where $a_1, \dots, a_n \in \mathbb{Z}^*$) be an admissible approximation of x belonging to \mathfrak{M} . We are going to define the measure of intelligence of (*); in particular, to know if it is intelligent or naive. To do so, we argue on the digits of real numbers in some basis of numeration. We use the decimal numeration system but we shall see just after that (fortunately) our result is independent on the choose of that system. Our idea for defining the concept of the intelligence of an approximation is the following:

The approximation (*) is intelligent if each digit of each of the integers a_1, \dots, a_n contributes to generating at least one exact digit of x (before and after the decimal point) in the approximation (*). In other words, the approximation (*) is intelligent if the sum of the number of digits of a_1, \dots, a_n is at most equal to the number of exact digits of x (before and after the decimal point) that the approximation (*) gives.

If (*) is not intelligent, we say that it is naive.

Since the number of digits of a positive integer a is approximatively¹ equal to $\frac{\log a}{\log 10}$, then the sum of the number of digits of the integers a_1, \dots, a_n is approximatively equal to

$$\sum_{i=1}^n \frac{\log |a_i|}{\log 10} = \frac{\log |a_1 \cdots a_n|}{\log 10}.$$

Remark that, up to a multiplicative constant (which is not significant), this last number is the logarithmic size of the approximation (*).

Similarly, the number of digits before the decimal point of x (i.e., the number of digits of the integer part of $|x|$) is approximatively¹ equal to

$$\frac{\log |x|}{\log 10}.$$

Further, the number of the exact digits after the decimal point that the approximation $(*)$ of x gives is approximatively¹ equal to

$$- \frac{\log |x - \mathfrak{M}(a_1, \dots, a_n)|}{\log 10}.$$

So, according to our preceding point of view of the intelligence of an approximation, the approximation $(*)$ is intelligent if and only if we have:

$$\frac{\log |a_1 \cdots a_n|}{\log 10} \leq \frac{\log |x|}{\log 10} - \frac{\log |x - \mathfrak{M}(a_1, \dots, a_n)|}{\log 10};$$

that is if and only if

$$\frac{\log |x| - \log |x - \mathfrak{M}(a_1, \dots, a_n)|}{\log |a_1 \cdots a_n|} \geq 1.$$

We remark in passing that this condition is independent on the choice of the decimal base. This leads us to propose the following definition:

Definition 4.1. Let \mathfrak{M} be a model of approximation of n parameters ($n \geq 1$) and let x be a nonzero real number not representable in \mathfrak{M} (i.e., $x \notin \mathfrak{M}(\mathbb{Z}^{*n})$). Let also

$$x \simeq \mathfrak{M}(a_1, \dots, a_n) \tag{*}$$

(where $a_1, \dots, a_n \in \mathbb{Z}^*$) be an admissible approximation of x in the model \mathfrak{M} .

We define the measure of intelligence of $(*)$, noted $\mu(*)$, by:

$$\boxed{\mu(*) := \frac{\log |x| - \log |x - \mathfrak{M}(a_1, \dots, a_n)|}{\log |a_1 \cdots a_n|}}.$$

We say that $(*)$ is intelligent if $\mu(*) \geq 1$ and we say that it is naive in the contrary.

Remark 4.2. It is clear from the preceding reasoning that more the measure of intelligence of an approximation is greater more this one is more intelligent. Thanks to this measure μ , we can henceforth compare between two approximations of a same real number x in a same model and say (without use the intuition) who is more intelligent than the other.

5 Examples

In what follows, we are going to give several examples of calculation of measures of intelligence of approximations. Note that each approximation is taken in the model which is both the finest

¹Actually, we can say that those approximations are exact if we adopt the philosophy of a non integer (resp. non positive) quantity of digits.

possible (i.e., the more general) and containing the minimum possible number of parameters. For example, the approximation $e \simeq 5\sqrt{2} - 3\sqrt{3} + 13\sqrt{6} - 31$ of e is taken in the model $a\sqrt{b}+c\sqrt{d}+f\sqrt{g}+h$; so neither in the model $a\sqrt{2}+b\sqrt{3}+c\sqrt{6}+d$ (because it is coarser than the previous one) and nor in the model $\frac{a\sqrt{b}+c\sqrt{d}+f\sqrt{g}+h}{k}$ (because the parameter k is supplementary).

5.1 Approximations concerning the number π

1. The Archimedes approximation of π is:

$$\pi \simeq \frac{22}{7} \quad (5.1)$$

The error of (5.1) is $|\pi - \frac{22}{7}| \simeq 1.2 \times 10^{-3}$ and its measure of intelligence (in the rational model) is:

$$\mu(5.1) = \frac{\log \pi - \log |\pi - \frac{22}{7}|}{\log(22 \times 7)} = 1.55 \dots \geq 1$$

This shows that Archimedes approximation of π is intelligent.

2. The well-known approximation:

$$\pi \simeq \frac{355}{113} \quad (5.2)$$

has the error $|\pi - \frac{355}{113}| \simeq 2.6 \times 10^{-7}$ and has the measure of intelligence (in the rational model):

$$\mu(5.2) = \frac{\log \pi - \log |\pi - \frac{355}{113}|}{\log(355 \times 113)} = 1.53 \dots \geq 1$$

So, that is an intelligent approximation but it is a little less intelligent than the previous one of Archimedes (although it is better in accuracy).

Note that those two previous approximations of π are both regular continued fraction convergents of π . In the next section, we will prove that any regular continued fraction convergent of a given nonzero real number is an intelligent approximation of that number in the rational model.

3. The well-known approximation:

$$\pi \simeq \sqrt{2} + \sqrt{3} \quad (5.3)$$

has the error $|\pi - (\sqrt{2} + \sqrt{3})| \simeq 4.6 \times 10^{-3}$ and has the measure of intelligence (in the model $\sqrt{a} + \sqrt{b}$):

$$\mu(5.3) = \frac{\log \pi - \log |\pi - (\sqrt{2} + \sqrt{3})|}{\log(2 \times 3)} = 3.63 \dots \geq 1$$

This shows that (5.3) is intelligent.

4. The well-known approximation:

$$\pi \simeq 2\sqrt{\sqrt{6}} \quad (5.4)$$

has the error $|\pi - 2\sqrt{\sqrt{6}}| \simeq 1.1 \times 10^{-2}$ and has the measure of intelligence (in the model $a\sqrt{\sqrt{b}}$):

$$\mu(5.4) = \frac{\log \pi - \log |\pi - 2\sqrt{\sqrt{6}}|}{\log(2 \times 6)} = 2.26 \dots \geq 1$$

So (5.4) is intelligent.

5. The well-known approximation:

$$\pi \simeq \frac{20}{9}\sqrt{2} \quad (5.5)$$

has the error $|\pi - \frac{20}{9}\sqrt{2}| \simeq 1.1 \times 10^{-3}$ and has the measure of intelligence (in the model $\frac{a}{b}\sqrt{c}$):

$$\mu(5.5) = \frac{\log \pi - \log |\pi - \frac{20}{9}\sqrt{2}|}{\log(20 \times 9 \times 2)} = 1.35 \dots \geq 1$$

So (5.5) is intelligent.

6. The well-known approximation:

$$\pi \simeq \frac{20}{11}\sqrt{3} \quad (5.6)$$

has the error $|\pi - \frac{20}{11}\sqrt{3}| \simeq 7.6 \times 10^{-3}$ and has the measure of intelligence (in the model $\frac{a}{b}\sqrt{c}$):

$$\mu(5.6) = \frac{\log \pi - \log |\pi - \frac{20}{11}\sqrt{3}|}{\log(20 \times 11 \times 3)} = 0.92 \dots < 1$$

So (5.6) is naive! But it is almost intelligent (because its measure of intelligence is quite close to 1).

7. Kochanski's approximation

$$\pi \simeq \sqrt{\frac{40}{3} - 2\sqrt{3}} \quad (5.7)$$

has the error $\left| \pi - \sqrt{\frac{40}{3} - 2\sqrt{3}} \right| \simeq 5.9 \times 10^{-5}$ and has the measure of intelligence (in the model $\sqrt{\frac{a}{b} + c\sqrt{d}}$):

$$\mu(5.7) = \frac{\log \pi - \log \left| \pi - \sqrt{\frac{40}{3} - 2\sqrt{3}} \right|}{\log(40 \times 3 \times 2 \times 3)} = 1.65 \dots \geq 1$$

So (5.7) is intelligent.

8. Ramanujan's approximation

$$\pi \simeq \frac{3}{5}(3 + \sqrt{5}) \quad (5.8)$$

has the error $|\pi - \frac{3}{5}(3 + \sqrt{5})| \simeq 4.8 \times 10^{-5}$ and has the measure of intelligence (in the model $\frac{a}{b}(c + \sqrt{d})$):

$$\mu(5.8) = \frac{\log \pi - \log |\pi - \frac{3}{5}(3 + \sqrt{5})|}{\log(3 \times 5 \times 3 \times 5)} = 2.04 \dots \geq 1$$

So (5.8) is intelligent.

The remaining approximations concerning the number π are all due to the author and they are all intelligent:

9. The approximation

$$\pi \simeq \sqrt{6(\sqrt{7}-1)} \quad (5.9)$$

has the error $\simeq 7.8 \times 10^{-4}$ and has the measure of intelligence $\simeq 2.22$. So it is an intelligent approximation of π .

10. The approximation

$$\pi \simeq 3 + \frac{1}{\sqrt{65}-1} \quad (5.10)$$

has the error $\simeq 5.1 \times 10^{-6}$ and has the measure of intelligence $\simeq 2.53$. So it is an intelligent approximation of π .

11. The approximation

$$\pi \simeq 3 + \frac{\sqrt{30}-1}{10\sqrt{10}} \quad (5.11)$$

has the error $\simeq 10^{-5}$ and has the measure of intelligence $\simeq 1.39$. So it is an intelligent approximation of π .

12. The approximation

$$\pi \simeq \frac{17}{11} \left(\frac{4}{\sqrt{15}} + 1 \right) \quad (5.12)$$

has the error $\simeq 4.8 \times 10^{-7}$ and has the measure of intelligence $\simeq 1.68$. So it is an intelligent approximation of π .

5.2 Approximations concerning the number e

1. The well-known rational approximation

$$e \simeq \frac{19}{7} \quad (5.13)$$

has the error $\simeq 4 \times 10^{-3}$ and has the measure of intelligence $\simeq 1.33$. So it is an intelligent approximation of e . Actually, the fraction $\frac{19}{7}$ is a regular continued fraction convergent of e .

The remaining approximations concerning the number e are all due to the author and they are all intelligent:

2. The approximation

$$e \simeq 3 - \frac{1}{3} \sqrt{\frac{5}{7}} \quad (5.14)$$

has the error $\simeq 8.6 \times 10^{-8}$ and has the measure of intelligence $\simeq 3$. So it is a “very” intelligent approximation of e . This approximation is very interesting because it is rare to find an approximation with this order of accuracy and with a measure of intelligence of this magnitude.

3. The approximation

$$e \simeq \frac{8}{3} + \frac{1}{11} \left(\frac{5}{2\sqrt{2}} - \frac{6}{5} \right)$$

has the error $\simeq 1.6 \times 10^{-8}$ and has the measure of intelligence $\simeq 1.58$. So it is an intelligent approximation of e .

4. The approximation

$$e \simeq \sqrt{4\sqrt{2} + \sqrt{3}}$$

has the error $\simeq 2.8 \times 10^{-5}$ and has the measure of intelligence $\simeq 3.6$. So it is a “very” intelligent approximation of e . We must note that the greatness of the measure of intelligence of this approximation is due to its simplicity much more than its accuracy.

5. The approximation

$$e \simeq \frac{35 - \sqrt{26}}{11}$$

has the error $\simeq 1.1 \times 10^{-5}$ and has the measure of intelligence $\simeq 1.35$. So it is an intelligent approximation of e .

6. The approximation

$$e \simeq \frac{8\sqrt{3} - 2\sqrt{2} + 8}{7}$$

has the error $\simeq 9.3 \times 10^{-7}$ and has the measure of intelligence $\simeq 1.73$. So it is an intelligent approximation of e .

7. The approximation

$$e \simeq 5\sqrt{2} - 3\sqrt{3} + 13\sqrt{6} - 31$$

has the error $\simeq 2.2 \times 10^{-7}$ and has the measure of intelligence $\simeq 1.33$. So it is an intelligent approximation of e .

8. The approximation

$$e \simeq \frac{3\sqrt{5} - 2}{\sqrt{3}}$$

has the error $\simeq 9.8 \times 10^{-7}$ and has the measure of intelligence $\simeq 3.3$. So it is a “very” intelligent approximation of e .

In what follows, we list in a tabular form some interesting approximations of other numbers with their corresponding errors and measures of intelligence.

| The real number | The approximation | its error | its measure of intelligence |
|----------------------|--|-----------------------|-----------------------------|
| \sqrt{e} | $\sqrt{3} - \frac{1}{12}$ | 3.8×10^{-6} | 3.6 |
| $\sqrt{\pi}$ | $\frac{13\sqrt{7} - 14\sqrt{5}}{4} + 1$ | 1.1×10^{-8} | 1.86 |
| π | $1 + \sqrt{\frac{8}{5}\sqrt{2} - \sqrt{3} - 4\sqrt{5} + 13}$ | 2.6×10^{-8} | 1.68 |
| $\frac{e}{\pi}$ | $\frac{11}{5} \left(\sqrt{2} + \sqrt{7} - \frac{11}{3} \right)$ | 7.5×10^{-8} | 1.6 |
| $\sqrt{e^2 + \pi^2}$ | $4 + \frac{119}{11880} + \frac{\sqrt{3}}{12}$ | 9.3×10^{-13} | 1.52 |

Remarks:

1. The approximation (5.10) can be found by using the regular continued fraction expansion of π , which is $[3, 7, 15, 1, 292, 1, \dots]$. Indeed, from this last, we have:

$$\begin{aligned}
\pi &\simeq [3, 7, 15, 1] \\
&= [3, 7, 16] \\
&\simeq [3, 7, \overline{16}].
\end{aligned}$$

But $x := [3, 7, \overline{16}]$ is a positive quadratic number which can be easily explicated. Putting $y := \frac{1}{x-3} + 9$, we have $y = [\overline{16}]$, so y satisfies the equation $y = 16 + \frac{1}{y}$. Solving this, we obtain (since $y > 16$): $y = 8 + \sqrt{65}$, which then gives $x = 3 + \frac{1}{\sqrt{65}-1}$. Consequently, we obtain the required approximation $\pi \simeq 3 + \frac{1}{\sqrt{65}-1}$.

On the other hand, by remarking that $65 = 8^2 + 1^2$, the approximation (5.10) can be used to establish an easy geometric construction of a line of length “very” close to π by using only a ruler and a compass. This should improve the famous Kochanski’s geometric construction invented for the same purpose.

2. Similarly to the above point, the approximation (5.14) can also be found by using the regular continued fraction expansion of e , which is $e = [2, 1, 2, 1, 1, 4, 1, \dots, 1, 2n, 1, \dots]$. Indeed, using this last, we have:

$$e \simeq [2, 1, 2, 1, 1, \overline{4, 1, 1, 6, 1, 1}].$$

The calculations show that $[2, 1, 2, 1, 1, \overline{4, 1, 1, 6, 1, 1}] = 3 - \frac{1}{3}\sqrt{\frac{5}{7}}$, which gives the required approximation.

3. An algebraic transformation of a given approximation changes its measure of intelligence. So, if we apply such a transformation, we can pass from an intelligent approximation to another that is less so or even naive (or vice versa). For example, the approximation (5.14): $e \simeq 3 - \frac{1}{3}\sqrt{\frac{5}{7}}$, which has a measure of intelligence $\simeq 3$, can be algebraically transformed into

$$e \simeq 3 - \frac{\sqrt{35}}{21};$$

which has a measure of intelligence (in the model $a + \frac{\sqrt{b}}{c}$) $\simeq 2.24$, which is well lower than that of the previous one. For a rational approximation (of a given real number), it is obvious that it would be the most intelligent when it is represented by an irreducible fraction. So, the algebraic style of an approximation plays a vital role in this context.

6 The intelligence of the rational approximations of a given irrational number

In this section, we present a detailed study of the intelligent approximations of a given irrational number in the rational model. As we will see below, this study is ultimately connected with the diophantine approximations in \mathbb{R} . We shall prove that in the rational model, there is always infinitely many intelligent approximations of any given irrational number x . Particularly, we prove that any regular continued fraction convergent of x is an intelligent approximation of x . In addition, we prove that for some $x \in \mathbb{R} \setminus \mathbb{Q}$, there exist intelligent rational approximations of x , not belonging to the above category.

Proposition 6.1. *Let α be an irrational number and let $p, q \in \mathbb{Z}^*$ such that $|pq| \neq 1$. Then the rational approximation $\alpha \simeq \frac{p}{q}$ is intelligent (in the rational model) if and only if one of the two following equivalent inequalities holds:*

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{|\alpha|}{|pq|} \quad (6.1)$$

$$\left| \frac{1}{\alpha} - \frac{q}{p} \right| \leq \frac{1}{p^2} \quad (6.2)$$

Proof. This is an immediate consequence of the definition of an intelligent approximation of a real number in a given model. \square

From Proposition 6.1, we deduce the following important corollary:

Corollary 6.2. *Let α be an irrational number. Then there is an infinitely many intelligent rational approximations of α .*

Proof. By Dirichlet's diophantine approximation theorem (see [2, Chapter 1]), there exist infinitely many rational numbers $\frac{q}{p}$ ($p, q \in \mathbb{Z}^*$) such that:

$$\left| \frac{1}{\alpha} - \frac{q}{p} \right| \leq \frac{1}{p^2}.$$

It follows (according to Proposition 6.1) that for each of such rational numbers $\frac{q}{p}$, the rational approximation $\alpha \simeq \frac{p}{q}$ is intelligent. This confirms the required result of the corollary. \square

We have also the following effective corollary:

Corollary 6.3. *Let α be an irrational number. Then any regular continued fraction convergent of α is an intelligent approximation of α in the rational model.*

Proof. If $[a_0, a_1, \dots]$ is the regular continued fraction expansion of α , then the regular continued fraction expansion of $\frac{1}{\alpha}$ is clearly equal to

$$\begin{cases} [a_1, a_2, \dots] & \text{if } a_0 = 0 \\ [0, a_0, a_1, \dots] & \text{if } a_0 \neq 0 \end{cases}.$$

It follows from this fact that for any given regular continued fraction convergent $\frac{p}{q}$ of α , the fraction $\frac{q}{p}$ is a regular continued fraction convergent of $\frac{1}{\alpha}$. So, according to the well-known properties of the regular continued fraction convergents of a real number (see [3, Chapter 1]), we have:

$$\left| \frac{1}{\alpha} - \frac{q}{p} \right| \leq \frac{1}{p^2};$$

which concludes (according to Proposition 6.1) that the approximation $\alpha \simeq \frac{p}{q}$ is intelligent in the rational model. The corollary is proved. \square

A variant of the measure of intelligence μ

Proposition 6.1 suggests us to introduce a new measure of intelligence μ' which is quite close to μ and which is, perhaps, more practical and more significant for the rational model.

Definition 6.4. Let \mathfrak{M} be a model of approximation of n parameters ($n \in \mathbb{N}^*$) and x be a nonzero real number not representable in \mathfrak{M} . We define the measure of intelligence μ' of an admissible approximation $x \simeq \mathfrak{M}(a_1, \dots, a_n)$ (where $a_1, \dots, a_n \in \mathbb{Z}^*$) of x in the model \mathfrak{M} by:

$$\mu'(x \simeq \mathfrak{M}(a_1, \dots, a_n)) := \frac{\log |\mathfrak{M}(a_1, \dots, a_n)| - \log |\alpha - \mathfrak{M}(a_1, \dots, a_n)|}{\log |a_1 \cdots a_n|}.$$

If $\mu'(x \simeq \mathfrak{M}(a_1, \dots, a_n)) \geq 1$, we say that the approximation $x \simeq \mathfrak{M}(a_1, \dots, a_n)$ is μ' -intelligent and otherwise we say that it is μ' -naive.

The following proposition can be easily checked.

Proposition 6.5. *Let α be an irrational number and let $p, q \in \mathbb{Z}^*$ such that $|pq| \neq 1$. Then the approximation $\alpha \simeq \frac{p}{q}$ is μ' -intelligent in the rational model if and only if we have:*

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{q^2}. \quad \square$$

Using Dirichlet's diophantine approximation theorem (see [2, Chapter 1]) and the properties of the regular continued fraction convergents of a real number (see [3, Chapter 1]), we immediately deduce (from Proposition 6.5) the following corollary:

Corollary 6.6. *Let α be an irrational number. Then there is an infinitely many μ' -intelligent rational approximations of α . Particularly, any regular continued fraction convergent of α is a μ' -intelligent approximation of α in the rational model.* \square

For all that follows, we forget the measure μ' and we work only with our first measure of intelligence μ .

The following theorem shows the existence of intelligent rational approximations of some irrational numbers which are not regular continued fraction convergents of those numbers.

Theorem 6.7. *Let $x > 1$ be an irrational number and let $[a_0, a_1, \dots]$ be its regular continued fraction expansion (where $a_i \in \mathbb{N}^*$ for all $i \in \mathbb{N}$). Suppose that for some $n \in \mathbb{N}$, we have:*

$$a_{n+1} \geq a_n - 1 \geq 1.$$

Then the rational approximation

$$x \simeq [a_0, a_1, \dots, a_{n-1}, a_n - 1]$$

is intelligent but it is not a regular continued fraction convergent of x .

Proof. It is obvious that $[a_0, a_1, \dots, a_{n-1}, a_n - 1]$ is not a regular continued fraction convergent of x . So, it remains to show that the rational approximation $x \simeq [a_0, a_1, \dots, a_{n-1}, a_n - 1]$ is intelligent. To do so, we distinguish the two following cases, where the first one is almost trivial.

• If $n = 0$: (trivial case)

In this case, we have to show that the approximation $x \simeq a_0 - 1$ is intelligent. We have:

$$x = a_0 + \frac{1}{a_1 + \dots} \leq a_0 + \frac{1}{a_1} \leq a_0 + \frac{1}{a_0 - 1}$$

(because in this case, we have by hypothesis $a_1 \geq a_0 - 1$). It follows that:

$$|x - (a_0 - 1)| \leq 1 + \frac{1}{a_0 - 1} = \frac{a_0}{a_0 - 1} < \frac{x}{a_0 - 1} \quad (\text{because } x > a_0),$$

which confirms (according to Proposition 6.1) that $x \simeq a_0 - 1$ is an intelligent approximation of x (in the rational model).

• If $n \geq 1$:

Let $\frac{p}{q}$ and $\frac{p'}{q'}$ (where $p, q, p', q' \in \mathbb{N}^*$) be the irreducible rational fractions defined by:

$$\begin{aligned} \frac{p}{q} &:= [a_0, a_1, \dots, a_{n-1}] \\ \frac{p'}{q'} &:= [a_0, a_1, \dots, a_{n-1}, a_n - 1]. \end{aligned}$$

Setting

$$y := [0, 1, a_{n+1}, a_{n+2}, \dots] = \frac{1}{1 + \frac{1}{a_{n+1} + \frac{1}{a_{n+2} + \dots}}},$$

we can express x in terms of y as follows:

$$\begin{aligned} x &:= [a_0, a_1, \dots] \\ &= [a_0, a_1, \dots, a_{n-1}, [a_n, a_{n+1}, \dots]] \\ &= [a_0, a_1, \dots, a_{n-1}, a_n - 1, [0, 1, a_{n+1}, a_{n+2}, \dots]] \\ &= [a_0, a_1, \dots, a_{n-1}, a_n - 1, y] \end{aligned}$$

It follows, according to the elementary properties of the regular continued fractions (see [3, Chapter 1]), that:

$$x = \frac{p + p'y}{q + q'y} \quad (6.3)$$

Using this last, we have:

$$\left| x - \frac{p'}{q'} \right| = \left| \frac{p + p'y}{q + q'y} - \frac{p'}{q'} \right| = \frac{|pq' - p'q|}{q'(q + q'y)} = \frac{1}{q'(q + q'y)}$$

(because $|pq' - p'q| = 1$, since $\frac{p}{q}$ and $\frac{p'}{q'}$ are two consecutive convergents of a real number). Consequently, we have (according to (6.3)):

$$\left| x - \frac{p'}{q'} \right| = \frac{x}{q'(p + p'y)} \quad (6.4)$$

On the other hand, from the definition of y and the hypothesis of the theorem, we have:

$$y > \frac{1}{1 + \frac{1}{a_{n+1}}} \geq \frac{1}{1 + \frac{1}{a_n - 1}} = 1 - \frac{1}{a_n} \quad (6.5)$$

Next, by setting $\frac{p_{n-2}}{q_{n-2}}$ (where $p_{n-2}, q_{n-2} \in \mathbb{N}$) the irreducible rational fraction which is equal to $[a_0, a_1, \dots, a_{n-2}]$ (with the convention $(p_{n-2}, q_{n-2}) = (1, 0)$ if $n = 1$), we have (according to the elementary properties of the regular continued fractions):

$$p' = (a_n - 1)p + p_{n-2} \leq (a_n - 1)p + p = a_n p \quad (\text{since } p_{n-2} \leq p).$$

Thus

$$\frac{p}{p'} \geq \frac{1}{a_n} > 1 - y \quad (\text{according to (6.5)}).$$

Hence $\frac{p}{p'} + y > 1$; that is:

$$p + p'y > p'.$$

By inserting this in (6.4), we finally obtain:

$$\left| x - \frac{p'}{q'} \right| \leq \frac{x}{p'q'},$$

which shows (according to Proposition 6.1) that the rational approximation $x \simeq \frac{p'}{q'}$ $= [a_0, a_1, \dots, a_{n-1}, a_n - 1]$ is intelligent. This completes the proof of the theorem. \square

From the previous theorem, we deduce the following important corollary:

Corollary 6.8. *Let $x > 1$ be an irrational number whose regular continued fraction expansion contains a finite number of 1's. Then there exist infinitely many intelligent rational approximations of x which are not regular continued fraction convergents of x .*

Proof. Let $[a_0, a_1, \dots]$ be the regular continued fraction expansion of x . By hypothesis, we have $a_n \geq 2$ for any sufficiently large n . On the other hand, since $(a_n)_{n \in \mathbb{N}}$ is a sequence of positive integers, we have for infinitely many $n \in \mathbb{N}$: $a_{n+1} \geq a_n - 1$. Consequently, we have for infinitely many $n \in \mathbb{N}$:

$$a_{n+1} \geq a_n - 1 \geq 1.$$

It follows from Theorem 6.7 that for any such n , the rational approximation $x \simeq [a_0, a_1, \dots, a_{n-1}, a_n - 1]$ is intelligent but it is not a regular continued fraction convergent of x . This confirms the corollary. \square

The following theorem provides another category of intelligent rational approximations of some irrational numbers that don't appear in their regular continued fraction convergents.

Theorem 6.9. *Let $x > 1$ be an irrational number and let $[a_0, a_1, \dots]$ be its regular continued fraction expansion. Suppose that for some $n \in \mathbb{N}$, we have:*

$$2 \leq a_{n+1} \leq a_n + 1.$$

Then the rational approximation $x \simeq [a_0, a_1, \dots, a_{n-1}, a_n + 1]$ is intelligent but it is not a regular continued fraction convergent of x . \square

Proof. Because $[a_0, a_1, \dots, a_{n-1}, a_n + 1] = [a_0, a_1, \dots, a_{n-1}, a_n, 1]$ and $a_{n+1} \geq 2$, then $[a_0, a_1, \dots, a_{n-1}, a_n + 1]$ is not a regular continued fraction convergent of x . Now, let us show that the rational approximation $x \simeq [a_0, a_1, \dots, a_{n-1}, a_n + 1]$ is intelligent. To do so, we distinguish the two following cases, where the first one is almost trivial.

• If $n = 0$: (trivial case)

In this case, we have to show that the approximation $x \simeq a_0 + 1$ is intelligent. We have:

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} \geq a_0 + \frac{1}{a_1 + 1} \geq a_0 + \frac{1}{a_0 + 2}$$

(because in this case, we have by hypothesis $a_1 \leq a_0 + 1$). It follows that:

$$\begin{aligned} |x - (a_0 + 1)| &= a_0 + 1 - x \leq (a_0 + 1) - \left(a_0 + \frac{1}{a_0 + 2}\right) = \frac{a_0 + 1}{a_0 + 2} \\ &= \frac{1}{a_0 + 1} \left(a_0 + \frac{1}{a_0 + 2}\right) \\ &\leq \frac{x}{a_0 + 1}, \end{aligned}$$

which confirms (according to Proposition 6.1) that $x \simeq a_0 + 1$ is an intelligent approximation of x (in the rational model).

• If $n \geq 1$:

Let $\frac{p}{q}$ and $\frac{p'}{q'}$ (where $p, q, p', q' \in \mathbb{N}^*$) be the irreducible rational fractions defined by:

$$\begin{aligned}\frac{p}{q} &:= [a_0, a_1, \dots, a_{n-1}] \\ \frac{p'}{q'} &:= [a_0, a_1, \dots, a_{n-1}, a_n + 1]\end{aligned}$$

and let

$$y := [0, -1, a_{n+1}, a_{n+2}, \dots] = \frac{1}{-1 + \frac{1}{a_{n+1} + \frac{1}{a_{n+2} + \ddots}}}$$

(remark that $y < -1$). We can express x in terms of y as follows:

$$\begin{aligned}x &:= [a_0, a_1, \dots] \\ &= [a_0, a_1, \dots, a_{n-1}, a_n + 1, y].\end{aligned}$$

It follows, according to the elementary properties of the regular continued fractions (see [3, Chapter 1]), that:

$$x = \frac{p + p'y}{q + q'y} \tag{6.6}$$

Using this last, we have:

$$\left| x - \frac{p'}{q'} \right| = \left| \frac{p + p'y}{q + q'y} - \frac{p'}{q'} \right| = \frac{|pq' - p'q|}{q'|q + q'y|} = \frac{1}{q'|q + q'y|}$$

(because $|pq' - p'q| = 1$, since $\frac{p}{q}$ and $\frac{p'}{q'}$ are two consecutive convergents of a real number). Consequently, we have (according to (6.6)):

$$\left| x - \frac{p'}{q'} \right| = \frac{x}{q'|p + p'y|} \tag{6.7}$$

On the other hand, from the definition of y and the hypothesis of the theorem, we have:

$$1 + \frac{1}{y} = \frac{1}{a_{n+1} + \frac{1}{\ddots}} > \frac{1}{a_{n+1} + 1} \geq \frac{1}{a_n + 2},$$

which gives:

$$y < -1 - \frac{1}{a_n + 1} \tag{6.8}$$

Next, by setting $\frac{p_{n-2}}{q_{n-2}}$ (where $p_{n-2}, q_{n-2} \in \mathbb{N}$) the irreducible rational fraction which is equal to $[a_0, a_1, \dots, a_{n-2}]$ (with the convention $(p_{n-2}, q_{n-2}) = (1, 0)$ if $n = 1$), we have (according to the elementary properties of the regular continued fractions):

$$p' = (a_n + 1)p + p_{n-2} \geq (a_n + 1)p.$$

Thus

$$\frac{p}{p'} \leq \frac{1}{a_n + 1} < -y - 1 \quad (\text{according to (6.8)}).$$

Hence $\frac{p}{p'} + y < -1$; that is $p + p'y < -p'$. Thus

$$|p + p'y| > p'.$$

By inserting this in (6.7), we finally obtain:

$$\left| x - \frac{p'}{q'} \right| < \frac{x}{p'q'},$$

which shows (according to Proposition 6.1) that the rational approximation $x \simeq \frac{p'}{q'}$ $= [a_0, a_1, \dots, a_{n-1}, a_n + 1]$ is intelligent. This completes the proof of the theorem. \square

Applications of Theorems 6.7 and 6.9 for the numbers π and e and some others

For the number π

The first applications of Theorem 6.7 for $\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, \dots]$ show that the two rational approximations $\pi \simeq 2$ ($= [2]$) and $\pi \simeq \frac{19}{6}$ ($= [3, 6]$) are both intelligent but none of them is a regular continued fraction convergent of π . On the other hand, an application of Theorem 6.9 for π shows that the rational approximation $\pi \simeq \frac{521030}{165849}$ ($= [3, 7, 15, 1, 292, 1, 1, 2]$) is intelligent but it is not a regular continued fraction convergent of π .

For the number π , we propose the following open problem:

An open problem. Is there a finite or an infinite number of intelligent rational approximations of π that don't appear in its regular continued fraction convergents?

For the number e

For the number $e = [2, 1, 2, 1, 1, 4, 1, \dots, 1, 2k, 1, \dots]$, it is clear that Theorem 6.7 can be applied in only one way² which gives that the rational approximation $e \simeq \frac{5}{2}$ ($= [2, 1, 1]$) is intelligent but it is not a regular continued fraction convergent of e . Similarly, Theorem 6.9 can also be applied in only one way and gives the same result (since $[2, 2] = [2, 1, 1] = \frac{5}{2}$).

By calculations, we found that also the approximations $e \simeq \frac{38}{14}$ and $e \simeq \frac{386}{142}$ are intelligent but they respectively reduce to the two regular continued fraction convergents $\frac{19}{7}$ and $\frac{193}{71}$ of e . This leads us to propose the following conjecture:

²Note that the approximation $e \simeq 1$ (obtained from the application of Theorem 6.7 for $n = 0$) is inadmissible because its logarithmic size is zero.

Conjecture. The approximation $e \simeq \frac{5}{2}$ is the only intelligent rational approximation of e which cannot reduce to one of its regular continued fraction convergents.

For the number $\sqrt{2}$

The applications of Theorem 6.7 for $\sqrt{2} = [1, \overline{2}]$ show that for all $n \in \mathbb{N}$, the rational approximation $\sqrt{2} \simeq [1, \underbrace{2, 2, \dots, 2}_{n \text{ times}}, 1]$ is intelligent but it is not a regular continued fraction convergent of $\sqrt{2}$. Actually, it is easy to show that for any positive integer n , we have:

$$[1, \underbrace{2, 2, \dots, 2}_{n \text{ times}}, 1] = \frac{2}{[1, \underbrace{2, 2, \dots, 2}_{n \text{ times}}]}.$$

So for any regular continued fraction convergent r of $\sqrt{2}$, the rational approximation $\sqrt{2} \simeq \frac{2}{r}$ is intelligent but it is not a regular continued fraction convergent of $\sqrt{2}$. Consequently, the number $\sqrt{2}$ has infinitely many intelligent rational approximations outside its regular continued fraction convergents.

The applications of Theorem 6.9 for $\sqrt{2}$ essentially give the same results because we have for any $n \in \mathbb{N}$:

$$[1, \underbrace{2, 2, \dots, 2}_{n \text{ times}}, 3] = [1, \underbrace{2, 2, \dots, 2}_{(n+1) \text{ times}}, 1].$$

For the number $\sqrt{2}$, we propose the following conjecture:

Conjecture. Any intelligent rational approximation of $\sqrt{2}$ has one of the two forms: r_n or $\frac{2}{r_n}$ ($n \in \mathbb{N}$), where r_n denotes the n^{th} regular continued fraction convergent of $\sqrt{2}$.

For the number $\sqrt{5}$

Let $(F_n)_{n \in \mathbb{N}}$ and $(L_n)_{n \in \mathbb{N}}$ denote respectively the usual Fibonacci and Lucas sequences, which are defined by:

$$\left\{ \begin{array}{l} F_0 = 0, F_1 = 1 \\ F_{n+2} = F_n + F_{n+1} \quad (\forall n \in \mathbb{N}) \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} L_0 = 2, L_1 = 1 \\ L_{n+2} = L_n + L_{n+1} \quad (\forall n \in \mathbb{N}) \end{array} \right.$$

It is easy to show that we have $\gcd(F_n, L_n) = 2$ if $n \equiv 0 \pmod{3}$ and $\gcd(F_n, L_n) = 1$ if $n \not\equiv 0 \pmod{3}$. The known fact that $\lim_{n \rightarrow +\infty} \frac{L_n}{F_n} = \sqrt{5}$ shows that the fractions $\frac{L_n}{F_n}$ ($n \geq 2$) are rational approximations of the number $\sqrt{5}$. Besides, those rational approximations are reducible if and only if $n \equiv 0 \pmod{3}$. In that case, the fraction $\frac{L_n}{F_n}$ reduces to $\frac{L_n/2}{F_n/2}$ and then becomes simpler. Actually, the last fractions are precisely the regular continued fraction convergents of $\sqrt{5}$; that is, the regular continued fraction convergents of $\sqrt{5}$ are the fractions

$\frac{L_{3n}/2}{F_{3n}/2}$ ($n \geq 1$). On the other hand, by calculations, we can easily show that for any $n \in \mathbb{N}^*$, the approximation $\sqrt{5} \simeq \frac{L_n}{F_n}$ is intelligent (in the rational model); so the rational approximations $\sqrt{5} \simeq \frac{L_{3n+1}}{F_{3n+1}}$ ($n \geq 1$) and $\sqrt{5} \simeq \frac{L_{3n+2}}{F_{3n+2}}$ ($n \geq 1$) are all intelligent but none of them is a regular continued fraction convergent³ of $\sqrt{5}$. Consequently, the number $\sqrt{5}$ has an infinitely many intelligent rational approximations outside its regular continued fraction convergents. It is remarkable that we can arrive at the same results by using Theorems 6.7 and 6.9. Indeed, because we have $\sqrt{5} = [2, \overline{4}]$, Theorem 6.7 applies and shows that for any $n \in \mathbb{N}$, the rational approximation $\sqrt{5} \simeq [2, \underbrace{4, 4, \dots, 4}_{n \text{ times}}, 3] = \frac{L_{3n+4}}{F_{3n+4}}$ is intelligent but it is not a regular continued fraction convergent of $\sqrt{5}$. Also, Theorem 6.9 applies and shows that for any $n \in \mathbb{N}$, the rational approximation $\sqrt{5} \simeq [2, \underbrace{4, 4, \dots, 4}_{n \text{ times}}, 5] = \frac{L_{3n+5}}{F_{3n+5}}$ is intelligent but it is not a regular continued fraction convergent of $\sqrt{5}$.

For the number $\sqrt{5}$, we propose the following conjecture:

Conjecture: Any intelligent rational approximation of $\sqrt{5}$ has one of the two forms: $\frac{L_{3n}/2}{F_{3n}/2}$ or $\frac{L_n}{F_n}$ ($n \geq 1$).

Real numbers with bounded (resp. unbounded) measures of intelligence of their rational approximations

Irrational numbers are of different natures in terms of the intelligence of their rational approximations. An important class of them does not possess “very intelligent” rational approximations! The result below shows that the irrational numbers for which the set of the measure of intelligence of their rational approximations is unbounded are exactly the Liouville numbers. First, let us recall the definition of the Liouville numbers:

Definition 6.10 (see [2]). An irrational number x is called a *Liouville number* if for every positive integer n , there exist integers a, b , with $b > 1$, such that:

$$\left| x - \frac{a}{b} \right| < \frac{1}{b^n}.$$

We have the following:

Theorem 6.11. *Let x be an irrational number. Then the set of the measures of intelligence of all the (admissible) rational approximations of x is unbounded if and only if x is a Liouville number.*

Proof.

• Suppose that x is a Liouville number. Then for any positive integer n , there exist $a_n, b_n \in \mathbb{Z}^*$ (with $b_n \geq 2$) such that:

$$\left| x - \frac{a_n}{b_n} \right| < \frac{1}{b_n^n} \quad (6.9)$$

³We can show that the equations $\frac{L_{3n+1}}{F_{3n+1}} = \frac{L_{3m}}{F_{3m}}$ and $\frac{L_{3n+2}}{F_{3n+2}} = \frac{L_{3m}}{F_{3m}}$ are impossible for $n, m \in \mathbb{N}^*$.

From (6.9), we derive that for all $n > 1$, we have: $|\frac{a_n}{b_n}| < |x| + \frac{1}{b_n} < |x| + 1$ and then we have $\log |\frac{a_n}{b_n}| < \log(|x| + 1) \leq |x|$; hence

$$\log \left| \frac{a_n}{b_n} \right| < |x| \quad (6.10)$$

Next, using (6.9) and (6.10), we have:

$$\begin{aligned} \mu \left(x \simeq \frac{a_n}{b_n} \right) &= \frac{\log |x| - \log \left| x - \frac{a_n}{b_n} \right|}{\log |a_n b_n|} \\ &> \frac{\log |x| + n \log(b_n)}{\log |a_n b_n|} \quad (\text{according to (6.9)}) \\ &= \frac{\log |x| + n \log(b_n)}{\log |\frac{a_n}{b_n}| + 2 \log(b_n)} \\ &> \frac{\log |x| + n \log(b_n)}{|x| + 2 \log(b_n)} \quad (\text{according to (6.10)}) \\ &= \frac{n + \frac{\log |x|}{\log(b_n)}}{2 + \frac{|x|}{\log(b_n)}} \\ &\geq \frac{n + \frac{\log |x|}{\log(b_n)}}{2 + \frac{|x|}{\log 2}} \quad (\text{since } b_n \geq 2). \end{aligned}$$

But because $(\frac{\log |x|}{\log(b_n)})_n$ is a bounded sequence, we have $\lim_{n \rightarrow +\infty} \frac{n + \frac{\log |x|}{\log(b_n)}}{2 + \frac{|x|}{\log 2}} = +\infty$. Thus

$$\lim_{n \rightarrow +\infty} \mu \left(x \simeq \frac{a_n}{b_n} \right) = +\infty,$$

concluding that the set of the measures of intelligence of all the (admissible) rational approximations of x is unbounded.

• Reciprocally, suppose that x is not a Liouville number. Then there exists a positive integer k such that for any rational approximation $\frac{a}{b}$ of x (with $a, b \in \mathbb{Z}^*$ and $b \geq 2$), we have:

$$\left| x - \frac{a}{b} \right| \geq \frac{1}{b^k}.$$

This gives:

$$\begin{aligned} \mu \left(x \simeq \frac{a}{b} \right) &= \frac{\log |x| - \log \left| x - \frac{a}{b} \right|}{\log |ab|} \\ &\leq \frac{\log |x| + k \log b}{\log |ab|} \\ &= \frac{\log |x|}{\log |ab|} + \frac{k \log b}{\log |ab|} \\ &\leq \frac{|x|}{\log 2} + k, \end{aligned}$$

showing that any rational approximation of x , with denominator ≥ 2 , has a measure of intelligence bounded (above) by $(\frac{|x|}{\log 2} + k)$. On the other hand, it is immediate that the measure of intelligence of any rational approximation of x , with denominator 1 and numerator $\notin \{0, 1, -1\}$ (i.e., any admissible integer approximation of x), is bounded above by $\frac{|x| - \log d(x, \mathbb{Z})}{\log 2}$ (where d denotes the usual distance on \mathbb{R}). This implies the required result and completes this proof. \square

We derive from Theorem 6.11 the following corollary:

Corollary 6.12. *Let α be an irrational algebraic number. Then the set of the measures of intelligence of all the (admissible) rational approximations of α is bounded.*

Proof. By Liouville's diophantine approximation theorem (see [2, §1]), α is not a Liouville number. The required result then immediately follows from Theorem 6.11. \square

Remark: If an irrational real number x is not a Liouville number then there exist $\kappa > 0$ and $c = c(x, \kappa) > 0$ such that for any rational number $\frac{a}{b}$ ($a, b \in \mathbb{Z}$, $b > 0$), we have:

$$\left| x - \frac{a}{b} \right| \geq \frac{c(x, \kappa)}{b^\kappa}.$$

Such κ (not necessary integers) are called *irrationality measures* of x . Besides, if $c(x, \kappa)$ can be explicitly computed, then κ is called *an effective irrationality measure* of x .

Much of important transcendental numbers are known not to be Liouville numbers and irrationality measures of them are calculated. Among these numbers, we list: $\pi, e, \log 2, \log 3, \zeta(3)$ (see [1, §11.3, pp. 362-386], [4], [5], [6], [7]). Consequently, for each of these numbers, the set of the measures of intelligence of all its (admissible) rational approximations is bounded. In other words (roughly speaking):

Each of the numbers $\pi, e, \log 2, \log 3, \zeta(3)$ does not have a “very intelligent” rational approximation.

7 An important open problem

Given a model of approximation \mathfrak{M} , let $\overline{\mathfrak{M}}$ denote the closure of the image of \mathfrak{M} (with respect to the usual topology of \mathbb{R}). Numerical experiences lead us to believe that for any $x \in \overline{\mathfrak{M}} \setminus \mathfrak{M}$, there exist intelligent approximations of x in \mathfrak{M} . However, proving (or disproving) this assertion appears (in general) to be difficult. The success we have had for the rational model is (as seen above) a consequence of Dirichlet's approximation theorem, which is itself a consequence of Dirichlet's pigeonhole principle (see [2, §1]). Unfortunately, this elementary principle cannot be applied for other models. So we ask here if there exist other methods (based perhaps on the density in \mathbb{R} or on the distribution modulo 1 of real sequences) which permit to solve the problem even for particular cases of models.

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